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ON BIPARTITE GRAPHS WITH LINEAR RAMSEY NUMBERS

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Dedicated to the memory of Paul Erdős

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We provide an elementary proof of the fact that the ramsey number of every bipartite graph H with maximum degree at most Δ is less than $8(8\Delta)^{\Delta}|V(H)|$. This improves an old upper bound on the ramsey number of the n-cube due to Beck, and brings us closer toward the bound conjectured by Burr and Erdős. Applying the probabilistic method we also show that for all $\Delta \geq 1$ and $n \geq \Delta+1$ there exists a bipartite graph with n vertices and maximum degree at most Δ whose ramsey number is greater than $c^{\Delta}n$ for some absolute constant $c>1$.

1. Introduction

For any graph H, we will denote by $r(H)$ the least integer N such that in any 2-coloring of the edges of K_N, the complete graph on N vertices, some monochromatic copy of H must always be formed. The existence of $r(H)$ is guaranteed by the classic theorem of Ramsey, and indeed, we will refer to $r(H)$ as the *ramsey number* of H. For dense graphs H, $r(H)$ tends to grow exponentially in the size of H. For example, the extreme case of $H=K_n$ has $r(K_n)$ lying roughly between $2^{n/2}$ and 4^n (see [7] for more precise bounds).

However, for relatively sparse graphs, $r(H)$ grows much more modestly. One parameter which measures the density of a graph is its *degeneracy number* $\max_{H' \subseteq H} \delta(H')$, where $\delta(H)$ is the minimum degree in H. Low degeneracy number is equivalent to low average degree of all subgraphs. Burr

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and Erdős [2] conjectured that *for each Δ there exists a constant $c(\Delta)$ such that for all graphs H with the degeneracy number at most Δ, we have $r(H) \le c(\Delta)|V(H)|$.* This conjecture still remains unresolved.

A particular class of graphs for which the Burr–Erdős conjecture has been proved is the class of graphs H of maximum degree at most Δ. It was shown by Chvatál, Rödl, Szemerédi and Trotter [4] that for each Δ there exists a constant $c(\Delta)$ so that for all such graphs H we have $r(H) \le c(\Delta)|V(H)|$. That is, the ramsey numbers for these graphs grow *linearly* with their size. Unfortunately, the estimate for $c(\Delta)$ was very weak, since the proof in [4] used the powerful Regularity Lemma of Szemerédi [14] (it grew like an exponential tower of 2's of height Δ).

In [6] we dispensed with the Regularity Lemma altogether, and obtained a bound of the form $c(\Delta) < \Delta^{c\Delta \log \Delta}$ for a suitable constant $c > 0$. We also showed that for all n and Δ there are graphs H with n vertices and maximum degree at most Δ such that $r(H') > c^{\Delta} n$ for a fixed constant $c > 1$.

Our proof of upper bound in [6] becomes particularily simple when we restrict ourselves to bipartite graphs only. In fact, in that case we can drop the logarithmic factor in the exponent.

Theorem 1. *For all integers $\Delta \ge 1$, if H is a bipartite graph with maximum degree at most Δ, then $r(H) < 8(8\Delta)^{\Delta}|V(H)|$.*

In particular, this improves an old upper bound on the ramsey number of the n-cube due to Beck [1]. For the sake of completeness, in Section 2 we provide a concise, elementary proof of Theorem 1 which is only outlined in [6]. We hope that our approach can be further refined to yield a complete solution (in the affirmative) of the Burr–Erdős conjecture.

Let us mention that for bipartite graphs H with maximum degree at most Δ, a doubly exponential bound $r(H) < 2^{2^{c\Delta}}|V(H)|$ follows from different versions of the Regularity Lemma considered by Eaton ([5], Lemma 3.3) and Komlós (cf. [10], Corollary 7.6).

Another result indicating that the ramsey numbers of bipartite graphs tend to be smaller than for arbitrary graphs was obtained in [11]. It is proved there that for highly unbalanced bipartite graphs $H = (X,Y,E)$, i.e., for those with significantly more vertices in one vertex class, say with $|X| \le |Y|^{\gamma}$, where $0 < \gamma < 1$, and with the degree of every vertex in Y not bigger than Δ, we have $r(H) < 2^{c_{\gamma}\Delta}$, where $c_{\gamma} > 0$ is a constant which depends only on γ and tends to infinity as γ approaches 1.

The main goal of this paper is to show that, despite the above results, ramsey numbers of bipartite graphs with maximum degree Δ can be almost as large as for non-bipartite graphs. In particular, it implies that the upper bound from Theorem 1 is reasonably close to the best possible.

Theorem 2. *There exists a constant $c>1$ such that for all $\Delta \geq 1$ and all $n \geq \Delta+1$ (except for $\Delta=1$ and $n=2,3,5$), there exists a bipartite graph H with n vertices and maximum degree at most Δ which satisfies $r(H)>c^{\Delta}n$.*

In the three exceptional cases, for all graphs H we have $r(H)=n$, and clearly, the conclusion of Theorem 2 could not be true.

Theorem 2 was announced in [6], and indeed our proof originates there. We apply the probabilistic method twice: first to prove the existence of a suitably structured graph H (Lemma 3), then to show the existence of a 2-coloring of K_N with no monochromatic copy of H (Lemma 4). The entire proof is the content of Section 3.

2. The proof of Theorem 1

If G is a bipartite graph with vertex set $V=V_1 \cup V_2$ and $A \subseteq V_1$ and $B \subseteq V_2$ then $G[A,B]$ denotes the induced subgraph of G on $A \cup B$, $e_G(A,B)$ stands for its number of edges and the density of the pair (A,B) is defined by

$$d_G(A,B) = \frac{e_G(A,B)}{|A|\,|B|}.$$

We will say that G is (ρ,d) - dense if for all $A \subset V_1$ and $B \subset V_2$ with $|A| \geq \rho|V|$ and $|B| \geq \rho|V|$, we have $d_G(A,B) \geq d$. It follows by a simple averaging argument that if G is not (ρ,d) - dense, then there are sets $A \subset V_1$ and $B \subset V_2$ of order $|A|=|B|=\lfloor \rho|V| \rfloor$, with $d_G(A,B)<d$. Let us emphasize that this is a weaker notion than the standard ϵ-regularity of G. Indeed, every ϵ-regular graph with density $d_G(V_1,V_2)=d$ is $(\epsilon, d-\epsilon)-$*dense*.

Before going into details, a rough sketch of the proof is as follows. For convenience, we will fix a balanced bipartition of K_{2N} and color only the edges between the two vertex classes. For N large, let $E(K_{N,N})=G_R \cup G_B$ be any 2-coloring of the edges of the bipartite complete graph $K_{N,N}$. If the graph G_R on the set of *Red* edges is *not* (ρ,d) - dense for appropriate ρ and d, then G_R must have a large induced subgraph of reasonably small maximum degree. This will imply (by an easy graph packing result – see Lemma 1 below) that H and G_R can be packed edge-disjointly in $K_{N,N}$, i.e., there is a *Blue* copy of H in $K_{N,N}$. On the other hand, if G_R *is* (ρ,d) - dense then by a standard embedding technique (see Lemma 2 below) G_R must contain a copy of H, which of course, gives us a *Red* copy of H in $K_{N,N}$.

Given two bipartite graphs G and H, with $V(G)=V_1 \cup V_2$ and $V(H)=X_1 \cup X_2$, we say that H can be embedded into G if there is an injection

$f: V(H) \to V(G)$, satisfying $f(X_i) \subseteq V_i$, $i=1,2$, called an embedding, such that for every edge $xy \in E(H)$, we have $f(x)f(y) \in E(G)$.

The (bipartite) complement $\overline{G}$ of G is the graph with $V(\overline{G}) = V(G)$ and $xy \in E(\overline{G})$ if and only if $x \in V_1$, $y \in V_2$ and $xy \notin E(G)$. We say that there is *a packing* of G and H if there is an embedding of H into $\overline{G}$. The maximum degree of a graph G is denoted by $\Delta(G)$.

Lemma 1. *Let H and G be as above, with $|V_1| = |V_2| \geq 2|V(H)|$. If $\max_{H' \subseteq H} \delta(H') \leq \Delta$ and $\Delta(G) \leq |V(G)|/(4\Delta)$, then there exists a packing of H and G.*

Proof. Let us set $|V(H)| = n$ and order the vertices of H as $x_1, x_2, \ldots, x_n$ so that for each i, vertex x_i has at most Δ neighbors in the set $L_i = \{x_1, \ldots, x_i\}$. Assume we have already packed $H[L_i]$, $i < n$, and denote by f the partial packing. Let $x_{i+1} \in V_1$ and $y_1, \ldots, y_k$, $k \leq \Delta$, be the neighbors of x_{i+1} in H which belong to L_i. Their images $f(y_1), \ldots, f(y_k)$ have together at most $\Delta \cdot \Delta(G) \leq |V_1|/2$ neighbors in V_1. Since $|f(L_i) \cap V_1| \leq i < n$, there is at least one vertex in V_1 which can be taken as the image $f(x_{i+1})$. We repeat this procedure until the entire graph H is packed with G. ∎

The following lemma is a special (bipartite) case of Lemma 2 in [6]. Its proof is an adaptation of that from [4]. Since it can be simplified in the bipartite case, we provide it here for completeness.

Lemma 2. *For all integers $\Delta \geq 1$, and for all positive numbers C, ρ and d such that*

$$d^{\Delta} \geq \rho\,, \qquad d^{\Delta} C \geq 1\,, \qquad \text{and} \qquad (1 - \Delta\rho)C \geq 1, \tag{1}$$

the following holds. If graphs H and G satisfy

(i) $\Delta(H) \leq \Delta$,

(ii) $|V_i| \geq C|V(H)|$, $i=1,2$ and

(iii) G is (ρ, d)-dense,

then H can be embedded into G.

Proof. We begin with an obvious consequence of the definition of a (ρ, d)-dense graph. Let $B \subseteq V_2$, $|B| \geq \rho |V_2|$. Call a vertex of V_1 *B-good* if it has at least $d|B|$ neighbors in B. Then at least $(1-\rho)|V_1|$ vertices of V_1 are B-good.

Set $|V_l| = N_l$ and $|X_l| = n_l$, $l=1,2$, for convenience. In particular, $n_1 + n_2 = n$ and $N_l \geq C n_l$, $l = 1,2$. Let us order the vertices of X_1 as $x_1, \ldots x_{n_1}$. We first sequentially embed the vertices of X_1 into V_1 in such a way that the following property is maintained throughout. For each $y \in X_2$ let Γ_y^i be the set of all neighbors of y within the set $L_i = \{x_1, \ldots x_i\}$, and, after embedding

L_i, let C_y^i be the set of those vertices of V_2 which are adjacent in G to every vertex of $f(\Gamma_y^i)$. We claim that $|C_y^i| \geq d^{|\Gamma_y^i|} N_2$ for all $y \in X_2$.

The proof of this claim is by induction on i. It is trivial for $i=0$ with the default setting $C_y^0 = V_2$. Let us assume that the claim is true after i steps. By the induction assumption and by (1), for each neighbor y of x_{i+1} we have $|C_y^i| \geq d^{|\Gamma_y^i|} N_2 \geq d^{\Delta} N_2 \geq \rho N_2$. Thus there are at least $(1-\rho\Delta)N_1 \geq n$ vertices of V_1 which are C_y^i-good for all neighbors y of x_{i+1} simultaneously. At least one of these vertices is outside the set $f(L_i)$ and we may select it as the image of x_{i+1}. Hence, after embedding all the vertices of X_1 we have, again by (1), $|C_y^{n_1}| \geq d^{\Delta} N_2 \geq n_2$ for all $y \in X_2$. We can now greedily map each $y \in X_2$ to a vertex from $C_y^{n_1}$, obtaining an embedding of H into G. ∎

Proof of Theorem 1. Let $E(K_{N,N}) = G_R \cup G_B$ be an arbitrary 2-coloring of the edges of $K_{N,N}$ where $N = Cn$ and $C = 4(8\Delta)^{\Delta}$. If G_R is $(\rho, 1/(8\Delta))$-dense, where $\rho = (8\Delta)^{-\Delta}$, then condition (1) is satisfied and, by Lemma 2, we can embed H into G_R, finding a *Red* copy of H.

Otherwise, there is a pair of sets $X \subset V_1$ and $Y \subset V_2$ such that $|X| = |Y| = \rho N = 4n$ and $d_{G_R}(X,Y) < 1/(8\Delta)$. Trivially, there are at most $2n$ vertices in X and at most $2n$ vertices in Y of degree greater than n/Δ. Removing the $2n$ largest degree vertices from each X and Y, we find subsets $X' \subset X$ and $Y' \subset Y$ such that $|X'| = |Y'| = \frac{1}{2}\rho N = 2n$ and $\Delta(G_R[X',Y']) \leq n/\Delta = |X'|/(2\Delta)$. By Lemma 1, we can find a packing of $G_R[X',Y']$ and H yielding a *Blue* copy of H. ∎

Remarks.

1. Our method of proof cannot give a better upper bound on $r(H)$ than the one obtained. Indeed, the packing lemma (Lemma 1) forces d to be of the order $1/\Delta$. Consequently, condition (1) of Lemma 2 requires that ρ be of the order $\Delta^{-\Delta}$ and we need $\rho N \geq 4n$. Note that the value of C is not crucial at all. In fact, we could use, instead of Lemma 2, the Blow-up Lemma (see [9], [12] and [13]) and reduce C to 1. However, every known proof of the Blow-up Lemma requires that ρ is of order d^{Δ}.

2. It might seem that a better tactic would be to take $d = 1/2$ and make the proof symmetric with respect to *Red* and *Blue*. This is indeed the original idea from [4] for which one first needs to secure a ρ-regular pair in, say G_R (if it happens to be sparser than 1/2 we simply switch to its blue complement which is also ρ-regular). However, even in Komlós' short-cut of the Regularity Lemma (where we are after just one regular pair) the size of that pair is roughly $2^{-3/\rho}N$, which, with ρ forced by (1) to be of the order at most $2^{-\Delta}$, gives only a doubly exponential bound. It is worthwhile

to mention that using Komlós' approach one can fix the majority color right from the beginning, thus obtaining a density statement. This result is implicit in [5].

3. Note also that the assumptions of Lemma 1 are weaker than $\Delta(H) \le \Delta$ and coincide with the assumptions of the Burr–Erdős conjecture. So, in a sense, we are "half-way" to its solution, at least in the bipartite case.

Finally, let us point out an interesting application of Theorem 1. Let Q_n be the n-cube. Then we have $|V(Q_n)| = 2^n$ and $\Delta = n$.

Corollary 1. *There is a constant $c > 0$ such that for all n, $r(Q_n) < 2^{cn \log n}$.* ∎

This improves an old result of Beck [1], who showed that $r(Q_n) < 2^{cn^2}$. Another conjecture of Burr and Erdős asserts that in fact $r(H) < c2^n$ (cf. [3]).

3. The proof of Theorem 2

In this section we prove Theorem 2. The proof rests on two lemmas, both proved by the probabilistic method with a random graph as the probability space. The random bipartite graph $G(n,n,M)$ is drawn uniformly from all bipartite graphs on $n+n$ labelled vertices and with M edges. The random graph $G(n,1/2)$ is a result of $\binom{n}{2}$ independent tosses of a fair coin, so its number of edges is a random variable with the binomial distribution $Bi(n,1/2)$. In this section partitions are allowed to have empty classes. Also, various expressions which do not look like integers should (usually) be rounded to the nearest corresponding integer.

Lemma 3. *There are fixed constants $c_0 > c_1 > 1$ and Δ_0 such that for each $\Delta \ge \Delta_0$ and $n \ge k^2$, where $k = c_0^{\Delta}$, there exists a bipartite graph H with vertex classes X and Y, $|X| = |Y| = n$, and with $\Delta(H) \le \Delta$, for which the following property holds. For all partitions $X = X_1 \cup \ldots \cup X_k$ and $Y = Y_1 \cup \ldots \cup Y_k$ with $|X_i|, |Y_i| \le (c_1/c_0)^{\Delta} n$, $i = 1, \ldots, k$, we have*

$$\sum_{i \ne j : e_H(X_i, Y_j) > 0} |X_i|\,|Y_j| > 0.55 n^2 \ . \tag{2}$$

Proof. Take any $1 < c_1 < c_0 < (10/7)^{1/202}$ and choose Δ_0 so that $(c_1/c_0)^{\Delta_0} < 0.1$ and $\left((0.7)^{1/101} c_0^2\right)^{\Delta_0} < 1/8$.

Let $\Delta \ge \Delta_0$ and $d = \Delta/101$. Consider the random bipartite graph $G(m,m,dm)$, where $m = 1.01n$, and denote its vertex classes (sides) by

V' and V''. Clearly, the number of vertices of degree larger than Δ in any bipartite graph with $m+m$ vertices and dm edges is, on each side, at most

$$\frac{dm}{\Delta+1} < \frac{m}{101} \ .$$

Thus, we will form the graph H by deleting from each side of $G(m,m,dm)$ the $n/100$ largest degree vertices so that $|V(H)|=2n$ and $\Delta(H)\le\Delta$.

We claim that, with positive probability, $G(m,m,dm)$ satisfies the following property: for all partitions $V'=V_1'\cup\ldots\cup V_k'\cup D'$ and $V''=V_1''\cup\ldots\cup V_k''\cup D''$ with $|V_i'|,|V_i''|\le(c_1/c_0)^{\Delta}n$, $i=1,\ldots,k$, and with $|D'|=|D''|=n/100$, we have

$$\sum_{i\neq j: e_H(V_i',V_j'')>0} |V_i'|\left|V_j''\right| > 0.55n^2 \ . \tag{3}$$

Indeed, since

$$\sum_i |V_i'||V_i''| \le (c_1/c_0)^{\Delta_0} n^2 < 0.1n^2 \ ,$$

any partition that violates (3) must satisfy

$$\sum_{i\neq j: e_G(V_i',V_j'')=0} |V_i'|\left|V_j''\right| \ge 0.35n^2 \ge 0.3m^2. \tag{4}$$

However, the expected number of partitions satisfying (4) is smaller than

$$(k+1)^{2m}2^{k^2}\frac{\binom{0.7m^2}{dm}}{\binom{m^2}{dm}} < 2^{k^2}(2k)^{2m}(0.7)^{dm} < 8^m\left((0.7)^{1/101}c_0^2\right)^{\Delta_0 m} < 1. \tag{5}$$

Above, the term $(k+1)^{2m}$ bounds the number of partitions, 2^{k^2} bounds the number of choices of the pairs (V_i',V_j'') for which $e_G(V_i',V_j'')$ is to be 0, while the fraction is an upper bound on the probability that no edge of $G(m,m,dm)$ will fall between these pairs.

Hence, there exists a graph $G\in G(m,m,dm)$ for which (3) holds. Setting D' to be the set of the $n/100$ largest degree vertices in V', and D'' to be the set of the $n/100$ largest degree vertices in V'', the graph $H=G-(D'\cup D'')$ fulfills the hypothesis of Lemma 4. ∎

The next lemma is a bipartite version of Lemma 5 in [6]. It is based on the simple fact that in $G(k,1/2)$, with high probability, there is about the expected number of edges between every pair of sufficiently large subsets of vertices.

Lemma 4. *For every $k \geq 2$, there exists a graph R on the vertex set $[k] = \{1,2,\ldots,k\}$ such that for all pairs of weight functions $f,g:[k] \to [0,1]$ with $f+g \leq 1$ and $\sum_{i=1}^k [f(i)+g(i)] = 2x > 10^8 \log k$, we have*

$$W = \sum_{ij \in R} [f(i)g(j) + f(j)g(i)] < 0.51x^2 \qquad \textit{and}$$

$$\overline{W} = \sum_{ij \notin R} [f(i)g(j) + f(j)g(i)] < 0.51x^2.$$

Proof. First observe that for any graph R and any fixed x, the quantity W is maximized by assignments f and g such that

$$f(i), g(j) \in \{0,1\} \tag{6}$$

for all i and j except for at most one vertex i_0 and one vertex j_0, where $i_0 \neq j_0$.

To see this, suppose first that for some f,g and i we have $0 < f(i), g(i) < 1$ and compare the sums $W_f(i) = \sum f(l)$ and $W_g(i) = \sum g(l)$ taken over all neighbors l of i in R. If $W_f(i) \geq W_g(i)$, define new functions f' and g' such that $f'(i) = 0$, $g'(i) = g(i) + f(i)$, and $(f'(l), g'(l)) = (f(l), g(l))$ for all $l \neq i$. Then the corresponding quantity W' is at least as large as W. If $W_f(i) < W_g(i)$, set $g'(i) = 0$ and $f'(i) = f(i) + g(i)$, again obtaining $W' \geq W$.

Thus, we may assume that $\min\{f(i), g(i)\} = 0$ for every i. If there exist i and j, $i \neq j$, such that $0 < g(i), g(j) < 1$, then compare $W_f(i)$ with $W_f(j)$. If $W_f(i) \geq W_f(j)$, set $\epsilon_{ij} = \min\{g(j), 1-g(i)\}$, $g'(i) = g(i) + \epsilon_{ij}$, and $g'(j) = g(j) - \epsilon_{ij}$. If $W_f(i) < W_f(j)$, do the same but with i and j swapped. In either case $W' \geq W$. Similarly, one shows that there is at most one vertex i_0 for which $0 < f(i_0) < 1$.

Set $T = \{i : f(i) = 1\}$ and $S = \{i : g(i) = 1\}$, and denote $t = |T|$ and $s = |S|$. Note that $t+s \leq 2x < t+s+2$ and, obviously, $2x \leq k$. We may further assume that $k > 10^8 \log 2$, since otherwise the condition $2x > 10^8 \log k$ would not be satisfied.

If (6) held for all i and j, we would have $W \leq e_R(T,S)$. With the two possible exceptions (i_0 and j_0) we still have $W \leq e_R(T,S) + 2x$. Suppose that $W \geq 0.51x^2$. Then, since x is large, $e_R(T,S) > 0.501x^2 \geq 0.501(t+s)^2/4$. This is, however, unlikely for the random graph $G(k,1/2)$ as the following estimate shows.

Assume that $t \leq s$ and note that when $t \leq s/7$, then $e_R(T,S) \leq ts < 0.5(t+s)^2/4$. Note also that $s > s_0 = 2 \cdot 10^7 \log k$. Hence, the probability that there exists a pair of such sets S and T is not greater than

$$\sum_{s > s_0} \sum_{s/7 \leq t < k-s} \binom{k}{s} \binom{k}{t} P(B > 0.501(t+s)^2/4),$$

where B is a random variable with the binomial distribution $Bi(ts,1/2)$. Since $t+s\le k$, we routinely bound $\binom{k}{s}\binom{k}{t}\le\binom{k}{s}^2<(ek/s)^{2s}$. As $(t+s)^2/4\ge ts$ and $t\ge s/7$, Chernoff's inequality (see e.g. [8], Remark 2.5) yields

$$P(B>0.501(t+s)^2/4)\le P(B>0.501ts)\le e^{-2\cdot 10^{-6}ts}<e^{-10^{-7}s^2}.$$

Altogether, the above probability is smaller than

$$\sum_{s>s_0}\sum_{s/7\le t<k-s}\left[(ek/s)^2e^{-10^{-7}s}\right]^s\le k\sum_{s>s_0}[e^2/s^2]^s<k^2[e^2/s_0^2]^{s_0}<1/2.$$

A similar argument establishes that also with probability greater than 1/2, the random graph $G(k,1/2)$ satisfies $\overline{W}<0.51x^2$ for all pairs of functions f and g as in the lemma. Hence, the required graph R exists. ∎

Now, to complete the proof of Theorem 2, we proceed as follows. Let c_0, c_1 and Δ_0 be as in the proof of Lemma 3 but with the additional requirement that $(c_0/c_1)^{\Delta_0}>10^8\Delta_0\log_2 c_0$. Also choose $c_2>1$ such that $c_2^{\Delta_0}<1.1$, and set $c_3=2^{1/4}/c_0^2$. Note that $c_3>1$. We will show that Theorem 2 holds with $c=\min\{c_1,c_2,c_3\}$.

If $1\le\Delta<\Delta_0$ and n is even, simply take H to be a matching. Then $r(H)=3n/2-1>1.1n>c_2^{\Delta_0}n>c^{\Delta}n$ for $n>2$. In the same case but with odd n, take H to be a matching plus one isolated vertex, giving $r(H)=3n/2-5/2>1.1n>c_2^{\Delta_0}n>c^{\Delta}n$ for $n>6$. When $\Delta\ge 2$ and $n=5$ (the case $n=3$ is impossible), take H to be the 4-cycle C_4 plus one isolated vertex. Then $r(H)=6>1.1n>c^{\Delta}n$.

If $\Delta\ge\Delta_0$ and $\Delta+1\le n<2c_0^{2\Delta}$, take H to be the complete bipartite graph $K_{\lfloor\Delta/2\rfloor,\lceil\Delta/2\rceil}$ plus $n-\Delta$ isolated vertices. Then

$$r(H)\ge r(K_{\lfloor\Delta/2\rfloor,\lfloor\Delta/2\rfloor})>2\cdot 2^{\Delta/4}=2c_3^{\Delta}c_0^{2\Delta}\ge c^{\Delta}n\ .$$

Finally, let us consider the main case when $\Delta\ge\Delta_0$ and $n\ge 2c_0^{2\Delta}$. Assume that n is even and set $q=n/2$ (if n is odd, we take the appropriate graph with $n-1$ vertices, supplemented by one isolated vertex). Choose H as in Lemma 3, but with q in place of n, and R as in Lemma 4. Use R to 2-color the edges of K_N, $N=c_1^{\Delta}n$, as follows. Partition arbitrarily $[N]=V(K_N)=U_1\cup\ldots\cup U_k$, $|U_i|=N/k$, $i=1,\ldots,k$, $k=c_0^{\Delta}$. Then for all $e\in[N]^2$, assign the color

$$\chi(e)=\begin{cases}\textit{Red}, & \text{if } e\in(U_i,U_j), ij\in R, i\ne j\\ \textit{Blue}, & \text{if } e\in(U_i,U_j), ij\notin R, i\ne j\\ \textit{arbitrary}, & \text{otherwise.}\end{cases}$$

We claim this coloring does not have a monochromatic copy of H. For suppose there is a *Red* copy H_0 of H, with vertex classes X and Y. Setting $X_i = X \cap U_i$ and $Y_i = Y \cap U_i$, we have by Lemma 3 that

$$\sum_{ij \in R} \{|X_i||Y_j| + |X_j||Y_i|\} \geq \sum_{i \neq j : e_{H_0}(X_i, Y_j) > 0} |X_i|\,|Y_j| > 0.55q^2 \tag{7}$$

On the other hand, expressing

$$|X_i| = f(i) \cdot N/k, \;\; i = 1, 2, \ldots, k \ ,$$

and

$$|Y_i| = g(i) \cdot N/k, \;\; i = 1, 2, \ldots, k \ ,$$

we have $0 \leq f(i) + g(i) \leq 1$, and

$$q = \sum_i |X_i| = \sum_i |Y_i| = N/k \sum_i f(i) = N/k \sum_i g(i) \ ,$$

so that

$$2x = \sum_i f(i) + \sum_i g(i) = kn/N = (c_0/c_1)^{\Delta} > 10^8 \log k$$

by our choice of c_0, c_1 and Δ_0, and by the monotonicity of $(c_0/c_1)^{\Delta}/\Delta$ as a function of Δ. Hence, by Lemma 4,

$$\begin{aligned}\sum_{ij \in R} \{|X_i||Y_j| + |X_j||Y_i|\} &= \frac{N^2}{k^2} \sum_{ij \in R} [f(i)g(j) + f(j)g(i)] \\ &< \frac{N^2}{k^2}(0.51)x^2 = 0.51q^2.\end{aligned}$$

This is a contradiction to (7), and the proof of Theorem 2 is complete. ∎

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